ISRAEL JOURNAL OF MATHEMATICS 134 (2003), 93–125

DENOMINATORS FOR THE POINCARÉ SERIES OF INVARIANTS OF MATRICES WITH INVOLUTION

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ABSTRACT

The goal of this paper is to study some Poincaré series associated to the invariants of the symplectic and odd orthogonal groups. These series turn out to be rational functions and our main results will describe the denominators. This work will generalize some known results on the invariants of the general linear groups. In addition to whatever intrinsic interest we hope our results may have, the subject involves an interesting interplay of invariant theory and complex variables.

* The first author gratefully acknowledges Support from DePaul University Research Council.

** The second author was supported in part by the Israel Science Foundation, founded by the Israel Academy of Sciences and Humanities, and by an Internal Research Grant from Bar-Ilan University.

Received January 10, 2002

Introduction

Let A be an element of the general linear group $\mathrm{GL}_n(\mathbb{C})$ and let $X_1, \ldots, X_k$ be $n \times n$ (generic) matrices with algebraically independent indeterminate entries $x^i_{\alpha,\beta}$, $\alpha, \beta = 1, \ldots, n$, $i = 1, \ldots, k$. Then there is an action of $\mathrm{GL}_n(\mathbb{C})$ on the polynomial ring $F = \mathbb{C}[x^i_{\alpha,\beta}]_{\alpha,\beta,i}$ defined by: $x^i_{\alpha,\beta}$ is mapped by A to the (α, β) entry of AX_iA^{-1}. The fixed ring of this action is denoted $\bar{C}$ and has a number of important properties. Let R be the algebra generated by the generic matrices $X_1, X_2, \ldots, X_k$. Then one of the important properties of $\bar{C}$ is that it is the algebra generated by the traces of elements of R. It is not hard to see that $\bar{C}$ has a k-fold grading by degree, and so there is associated to it a Poincaré series $P_k(\bar{C})$. It is known that $P_k(\bar{C})$ is a rational function, and it has been computed in a number of cases. One important tool has been the Weyl integration formula which expresses $P_k(\bar{C})$ as a complex integral

$$\frac{1}{n!} \oint_T \frac{\prod_{1 \leq \alpha \neq \beta \leq n} (1 - z_\alpha z_\beta^{-1})}{\prod_{i=1}^k \prod_{\alpha,\beta=1}^n (1 - z_\alpha z_\beta^{-1} t_i)} d\nu, \tag{1}$$

where the integral is over the torus $|z_\alpha| = 1$, for $\alpha = 1, \ldots, n$, and the measure is

$$d\nu = (2\pi i)^{-n} \frac{dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n}{z_1 z_2 \ldots z_n}.$$

We will say more about why this integral gives $P_k(\bar{C})$ in Section 1 in which we will generalize it to the other classical groups.

If we take the algebra $\bar{R} = R\bar{C}$ generated by generic matrices and their traces, we also get a ring of $\mathrm{GL}_n(\mathbb{C})$ invariants. Consider the $n \times n$ matrices over $\mathbb{C}[x^i_{\alpha,\beta}]_{\alpha,\beta,i}$ with $\mathrm{GL}_n(\mathbb{C})$ action obtained by the composition of the action of $\mathrm{GL}_n(\mathbb{C})$ defined above with conjugation on matrices. Then the fixed ring is $\bar{R}$ and so it again follows from Weyl's formula that $\bar{R}$ has Poincaré series which can be calculated as an integral. The integral is

$$\frac{1}{n!} \oint_T \frac{\prod_{1 \leq \alpha \neq \beta \leq n} (1 - z_\alpha z_\beta^{-1}) \cdot \sum_{\alpha,\beta=1}^n z_\alpha z_\beta^{-1}}{\prod_{i=1}^k \prod_{\alpha,\beta=1}^n (1 - z_\alpha z_\beta^{-1} t_i)} d\nu. \tag{2}$$

In [T1], [T2] and [BS], (1) and (2) were used to calculate the Poincaré series for $\bar{C}$ and $\bar{R}$ for $(n, k) = (2, 2)$, $(2, 3)$, $(3, 2)$, $(3, 3)$ and $(4, 2)$. (The case of 2×2 matrices was also done in [F1], [F2] and [L] using more purely algebraic methods.) In [BS], the least denominator was also calculated for $(3, k)$ and conjectured for $(4, k)$.

It is our goal in this paper to generalize this work from generic matrices to generic matrices with involution. For n odd the only involutions are of transpose type, and for n even the involution may be of either transpose or symplectic type. Procesi showed in [P] that the corresponding trace ring is a ring of invariants for either the orthogonal or symplectic group. It turns out in the symplectic and odd (but not even) orthogonal cases that the Poincaré series can be calculated using Weyl's integration formulas. Consider for example the case of the symplectic involution on $2n \times 2n$ matrices. The relevant integral is then

$$\frac{1}{2^n n!} \oint_T \frac{\prod_{1\le\alpha<\beta\le n}(1 - z_\alpha^{\pm1} z_\beta^{\pm1}) \cdot \prod_{\alpha=1}^n (1 - z_\alpha^{\pm2})}{\prod_{\alpha,\beta=1}^n \prod_{i=1}^k (1 - z_\alpha^{\pm1} z_\beta^{\pm1} t_i)} d\nu, \tag{3}$$

where T is the unit torus, as above.

Not only may $\bar{C}$ and $\bar{R}$ be considered as a k-fold graded algebra, but they can also be given a finer $2k$-fold grading. The algebra generated by k generic matrices together with their transposes is isomorphic to the algebra generated by k generic symmetric matrices and k generic skew symmetric matrices. These generators induce the $2k$-fold grading. More generally still, given an involution on matrices, we may consider the algebra generated by k_1 generic symmetric matrices and k_2 generic skew symmetric matrices. This algebra will have a $(k_1 + k_2)$-fold grading and a formal power series in variables $x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}$. We distinguish this series by referring to it as the *-Poincaré series. It turns out that the analogues of $\bar{C}$ and $\bar{R}$ in this case are also invariants for one of the classical groups, and that the integrals can be constructed to calculate the *-Poincaré series as well as the Poincaré series.

In Section 1 we present the results we need from invariant theory and construct the integrals which we will be studying. In Section 2 we use these integrals together with the algebraic properties of the generic matrix algebras to prove some general results about the Poincaré series and *-Poincaré series. These results are all known in the case of matrices without involution, and are mostly new in the present case. The main results of this section show that each series is a rational function and describes which type of terms can occur in the denominator (Theorem 6); give a functional equation satisfied by these rational functions in some cases (Theorem 9); and, identify the orders of the poles at 1 for the *-Poincaré series case, except for $x_i = 1$ in the symplectic case (Theorem 10).

In Sections 3 and 4, we investigate the case of 3×3 matrices with transpose involution. For convenience, we do the case of $\bar{C}$ and $\bar{R}$ separately. Our main results in these sections are that $P(\bar{C})$ can be written as a fraction in lowest terms with denominator

$$\prod_{i=1}^{k}(1-t_i)(1-t_i^2)^3(1-t_i^3)^2 \prod_{1\le i<j\le k}(1-t_it_j)^3(1-t_it_j^2)^2(1-t_i^2t_j)^2,$$

and that $P(\bar{R})$ can be written as a fraction in lowest terms with denominator

$$\prod_{i=1}^{k}(1-t_i)^3(1-t_i^2)^3 \prod_{1\le i<j\le k}(1-t_it_j)^3(1-t_it_j^2)^2(1-t_i^2t_j)^2.$$

We investigate least denominators for the *-Poincaré series, and calculate each of the series in a few cases. As a corollary, we are able to calculate the character sequences for $\bar{C}(0,k)$ and $\bar{R}(0,k)$.

In Section 5 we work on 4×4 matrices with symplectic involution. For each of the Poincaré series and *-Poincaré series we calculate a denominator. We suspect that it is not a least denominator, but is not too far off. We conjecture what we think the least denominator is in the *-Poincaré case.

1. Invariants and integrals

For more information on the classical groups we refer the reader to [FuH]. Let G be one of the groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n}(\mathbb{C})$ or $\mathrm{SO}_{2n+1}(\mathbb{C})$, and let M be a G-module. The group G contains a Cartan subgroup H, which we may take to be the set of all diagonal matrices in G. The character of M may be defined to be the trace of H on M, so the character will be a function of n variables. This character determines M up to G-isomorphism. In particular, it determines the multiplicity of each irreducible G-module in M and this multiplicity can be calculated from the character using integration. We will be interested only in the special case of the trivial character. As in the introduction, let $T\subset H$ be the diagonal elements with entries of absolute value 1 and let $d\nu$ be the translation-invariant measure

$$d\nu=\frac{1}{(2\pi i)^n}\frac{dz_1\wedge\cdots\wedge dz_n}{z_1\ldots z_n}$$

on T. Here is Weyl's integration formula.

THEOREM 1 (Weyl): *Given G, one of the classical group as above, there exists a polynomial $P_G=P(z_1,\ldots,z_n)$ such that if M is a G-module with character $f(z_1,\ldots,z_n)$, then $\oint_T Pf\,d\nu$ is the dimension of the space of G-invariant elements of M.*

We now record the polynomials P_G. Let

$$\Delta(z_1,\ldots,z_n):=\prod_{\alpha<\beta}(z_\beta-z_\alpha)$$

be the Vandermonde determinant. If $G = \mathrm{GL}_n(\mathbb{C})$ then

$$\begin{aligned} P(z_1,\ldots,z_n) =& \frac{1}{n!}\Delta(z_1,\ldots,z_n)\Delta(z_1^{-1},\ldots,z_n^{-1}) \\ =& \frac{1}{n!}\prod_{1\le\alpha\ne\beta\le n}(1-z_\alpha z_\beta^{-1}). \end{aligned}$$

If $G = \mathrm{Sp}_{2n}(\mathbb{C})$ then

$$\begin{aligned} P(z_1,\ldots,z_n) =& \frac{(-1)^n}{2^n n!}\Delta(z_1+z_1^{-1},\ldots,z_n+z_n^{-1})^2(z_1-z_1^{-1})^2\ldots(z_n-z_n^{-1})^2 \\ =& \frac{1}{2^n n!}\prod_{1\le\alpha<\beta\le n}(1-z_\alpha^{\pm1}z_\beta^{\pm1})\prod_{\alpha=1}^{n}(1-z_\alpha^{\pm2}), \end{aligned}$$

where we use the plus or minus notation, here and throughout, as a shorthand for $\prod_{\epsilon_1,\epsilon_2=\pm1}(1-z_a^{\epsilon_1}z_b^{\epsilon_2})$ in the first factor and $\prod_{\epsilon=\pm2}(1-z_a^{\epsilon})$ in the second.

If $G = \mathrm{SO}_{2n}(\mathbb{C})$ then

$$\begin{aligned} P(z_1,\ldots,z_n) =& \frac{1}{2^{n-1}n!}\Delta(z_1+z_1^{-1},\ldots,z_n+z_n^{-1})^2 \\ =& \frac{1}{2^n n!}\prod_{1\le\alpha<\beta\le n}(1-z_\alpha^{\pm1}z_\beta^{\pm1}). \end{aligned}$$

Finally, if $G = \mathrm{SO}_{2n+1}(\mathbb{C})$ then

$$\begin{aligned} P(z_1,\ldots,z_n) =& \frac{(-1)^n}{2^n n!}\Delta(z_1+z_1^{-1},\ldots,z_n+z_n^{-1})^2(z_1^{\frac12}-z_1^{-\frac12})^2\ldots(z_n^{\frac12}-z_n^{-\frac12})^2 \\ =& \frac{1}{2^n n!}\prod_{1\le\alpha<\beta\le n}(1-z_\alpha^{\pm1}z_\beta^{\pm1})\prod_{\alpha=1}^{n}(1-z_\alpha^{\pm1}). \end{aligned}$$

We will need Weyl's integration formula in the form of this corollary.

Corollary 2: *Let V be a vector space graded by $\mathbb{N}^n$,*

$$V = \bigoplus_{\alpha_1,\ldots,\alpha_n} V(\alpha_1,\ldots,\alpha_n)$$

such that each homogeneous subspace is finite dimensional. Assume that V is a G-module and that the action respects degree. Let $V(\alpha_1,\ldots,\alpha_n)$ have G-character $f_\alpha(z_1,\ldots,z_n)$ and let

$$F(t_1,\ldots,t_n) := \sum_\alpha f_\alpha(z_1,\ldots,z_n)t_1^{\alpha_1}\ldots t_n^{\alpha_n}.$$

Then the fixed ring V^G has Poincaré series equal to $\oint_T PFd\nu$.

Let $F = F(N,k)$ be the polynomial ring in the variables $x^i_{\alpha,\beta}$, $\alpha,\beta = 1,\ldots,N$, $i = 1,\ldots,k$ and recall the action of $\mathrm{GL}_N(\mathbb{C})$ on F in which $A \in \mathrm{GL}_N(\mathbb{C})$ sends each $x^i_{\alpha,\beta}$ to the (α,β) entry of the matrix AX_iA^{-1}, where X_i is the generic matrix with entries $x^i_{\alpha,\beta}$. There is an action of $\mathrm{GL}_N(\mathbb{C})$ on $M_N(F)$ obtained by composing the previous action on F with the conjugation $B \to A^{-1}BA$. The fixed ring $F^{GL_N(\mathbb{C})}$ is the ring $\bar{C} = \bar{C}(N,k)$ generated by the image of the trace map from the ring $R = R(N,k)$ generated by the generic matrices $\mathbb{C}[X_1,\ldots,X_k]$ to F, and the fixed ring $M_N(F)^{GL(\mathbb{C})}$ is the ring $\bar{R} = \bar{R}(N,k)$ generated by R and $\bar{C}$. These facts are due to Procesi, and are equivalent to the First Fundamental Theorem of invariant theory for the general linear group. If $N = 2n$ is even, then $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{O}_{2n}(\mathbb{C})$ are contained in $\mathrm{GL}_N(\mathbb{C})$ and if $N = 2n+1$ is odd, then $\mathrm{O}_{2n+1}(\mathbb{C})$ is contained in $\mathrm{GL}_N(\mathbb{C})$. So, by restriction, each classical group has an action on F and on $M_N(F)$.

Up to conjugation $M_N(\mathbb{C})$ accepts one involution if N is odd and two if N is even. These extend to $M_N(F)$. Consider the algebra generated by the generic matrices $X_1,\ldots,X_k$ together with their images under the involution, $X_1^*,\ldots,X_k^*$. If the involution is transpose type, we will denote this ring by $R(N,k;t)$ and if it is of symplectic type, by $R(N,k;s)$. The commutative algebras generated by the traces will be denoted $\bar{C}(N,k;t)$ and $\bar{C}(N,k;s)$, and the non-commutative algebras with trace generated by the generic matrices will be denoted $\bar{R}(N,k;t)$ and $\bar{R}(N,k;s)$. The next theorem is from Procesi's seminal paper [P].

THEOREM 3 (Procesi): *Let $F = F(N,k)$. Then*

(1) *if $N = n$, then $F^{GL_n(\mathbb{C})} = \bar{C}(n,k)$ and $M_n(F)^{GL_n(\mathbb{C})} = \bar{R}(n,k)$;*
(2) *if $N = 2n$, then $F^{O_{2n}(\mathbb{C})} = \bar{C}(2n,k;t)$ and $M_{2n}(F)^{O_n(\mathbb{C})} = \bar{R}(2n,k;t)$;*
(3) *if $N = 2n$, then $F^{Sp_{2n}(\mathbb{C})} = \bar{C}(2n,k;s)$ and $M_{2n}(F)^{Sp_{2n}(\mathbb{C})} = \bar{R}(2n,k;s)$;*
(4) *if $N = 2n+1$, then $F^{O_{2n+1}(\mathbb{C})} = \bar{C}(2n+1,k;t)$ and $M_{2n+1}(F)^{O_{2n+1}(\mathbb{C})} = \bar{R}(2n+1,k;t)$.*

Aside. Procesi also related each $\bar{C}$ and $\bar{R}$ to an algebra of simultaneous invariants of matrices. Consider a map $F\colon \mathrm{M}_n(\mathbb{C})^k \to \mathbb{C}$. Such a map will be called *G-invariant* if

$$F(gA_1g^{-1}, gA_2g^{-1},\ldots,gA_kg^{-1}) = F(A_1,A_2,\ldots,A_k)$$

for all $g \in G$ and all $A_1,\ldots,A_k \in \mathrm{M}_n(\mathbb{C})$. The set of all G invariant maps which are polynomial in the entries of the matrices forms an algebra, and Procesi's theorem says that this algebra is isomorphic to the appropriate $\bar{C}(n,k;-)$. If we instead consider the invariant maps from $\mathrm{M}_n(\mathbb{C})^k$ to $\mathrm{M}_n(\mathbb{C})$ we get an algebra isomorphic to $\bar{R}(n,k;-)$.

In (4) note that the odd orthogonal group $\mathrm{O}_{2n+1}(\mathbb{C})$ is the direct product of the special orthogonal group $\mathrm{SO}_{2n+1}(\mathbb{C})$ with the two element group generated by the scalar matrix $-I$. Since the action is by conjugation $-I$ acts as the identity so the $O_{2n+1}(\mathbb{C})$ invariants are the same as the $\mathrm{SO}_{2n+1}(\mathbb{C})$ invariants. In the case of $O_{2n}(\mathbb{C})$ we don't know of a similar reduction. Although the $\mathrm{SO}_{2n}(\mathbb{C})$ invariants may be of intrinsic interest, we choose not to compute them here because they don't relate directly to $\bar{C}(2n,k;t)$ or $\bar{R}(2n,k;t)$. Combining Theorem 3 with Corollary 2, we may express the Poincaré series of each $\bar{C}$ and $\bar{R}$, except for the even orthogonal case, as an integral, once we compute the character of each F and $M_N(F)$. The case of the general linear group was the subject of [V] and is included here only for the sake of completeness.

LEMMA 4: *Given n and k, the characters of $F = F(N,k)$ and $M_N(F)$ are given by*

(1) *F has $\mathrm{GL}_n(\mathbb{C})$-character*

$$\prod_{i=1}^{k}\prod_{\alpha,\beta=1}^{n}(1-z_\alpha z_\beta^{-1}t_i)^{-1}$$

and $M_n(F)$ has $\mathrm{GL}_n(\mathbb{C})$-character equal to the character of F times

$$\sum_{\alpha,\beta=1}^{n} z_\alpha z_\beta^{-1} = \Big(\sum_{\alpha=1}^{n} z_\alpha\Big)\Big(\sum_{\alpha=1}^{n} z_\alpha^{-1}\Big).$$

(2) *F has $\mathrm{Sp}_{2n}(\mathbb{C})$-character and $\mathrm{SO}_{2n}(\mathbb{C})$-character equal to*

$$\prod_{i=1}^{k}\prod_{\alpha,\beta=1}^{n}(1-z_\alpha^{\pm1}z_\beta^{\pm1}t_i)^{-1}$$

and $M_{2n}(F)$ has character equal to the character of F times

$$\Big(\sum_{\alpha=1}^{n}(z_\alpha+z_\alpha^{-1})\Big)^2.$$

(3) *F has $\mathrm{SO}_{2n+1}(\mathbb{C})$-character equal to*

$$\prod_{i=1}^{k}[\prod_{\alpha,\beta=1}^{n}(1-z_\alpha^{\pm1}z_\beta^{\pm1}t_i)^{-1}\prod_{\alpha=1}^{n}(1-z_\alpha^{\pm1}t_i)^{-2}(1-t_i^{-1})]$$

and $M_{2n+1}(F)$ has character equal to the character of F times

$$\left(1+\sum_{\alpha=1}^{n}(z_\alpha+z_\alpha^{-1})\right)^2.$$

Proof: Consider first the case of F. Let D be the diagonal matrix with entries $z_1,\ldots,z_n$ in the case of $\mathrm{GL}_n(\mathbb{C})$; entries $z_1,\ldots,z_n,z_1^{-1},\ldots,z_n^{-1}$ in the cases of $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n}(\mathbb{C})$; and with entries $z_1,\ldots,z_n,z_1^{-1},\ldots,z_n^{-1},1$ in the case of $\mathrm{SO}_{2n+1}(\mathbb{C})$. Each generator $x^i_{\alpha,\beta}$ of F is an eigenvector for D under the action. Computing the eigenvalues gives the characters of F as claimed. For $M_N(F)=M_N(\mathbb{C})\otimes F$ we multiply the character of F times the character of $M_N(\mathbb{C})$. In $M_N(\mathbb{C})$ each matrix unit $e_{\alpha,\beta}$ is an eigenvector and the result follows from an easy computation. ∎

We may now express each of the Poincaré series as a complex integral over the torus $|z_i|=1$, $i=1,\ldots,n$ with respect to the translation invariant measure $d\nu$.

$P(\bar{C}(2n,k;s))=$

$$\frac{1}{2^n n!}\oint_T \frac{\prod_{1\le\alpha<\beta\le n}(1-z_\alpha^{\pm1}z_\beta^{\pm1})\prod_{\alpha=1}^n(1-z_\alpha^{\pm2})}{\prod_{i=1}^k\prod_{\alpha,\beta=1}^n(1-z_\alpha^{\pm1}z_\beta^{\pm1}t_i)}d\nu, \tag{3}$$

$P(\bar{R}(2n,k;s))=$

$$\frac{1}{2^n n!}\oint_T \frac{\prod_{1\le\alpha<\beta\le n}(1-z_\alpha^{\pm1}z_\beta^{\pm1})\prod_{\alpha=1}^n(1-z_\alpha^{\pm2})(\sum_{\alpha,\beta=1}^n(z_\alpha+z_\alpha^{-1}))^2}{\prod_{i=1}^k\prod_{\alpha,\beta=1}^n(1-z_\alpha^{\pm1}z_\beta^{\pm1}t_i)}d\nu, \tag{4}$$

$P(\bar{C}(2n+1,k;t))=$

$$\frac{1}{2^n n!}\oint_T \frac{\prod_{1\le\alpha<\beta\le n}(1-z_\alpha^{\pm1}z_\beta^{\pm1})\prod_{\alpha=1}^n(1-z_\alpha^{\pm1})}{\prod_{i=1}^k(\prod_{\alpha,\beta=1}^n(1-z_\alpha^{\pm1}z_\beta^{\pm1}t_i)\prod_{\alpha=1}^n(1-z_\alpha^{\pm1}t_i)^2(1-t_i))}d\nu, \tag{5}$$

$P(\bar{R}(2n+1,k;t))=$

$$\frac{1}{2^n n!}\oint_T \frac{\prod_{1\le\alpha<\beta\le n}(1-z_\alpha^{\pm1}z_b^{\pm1})\prod_{\alpha=1}^n(1-z_\alpha^{\pm1})(1+\sum_{\alpha=1}^n(z_\alpha+z_\alpha^{-1}))^2}{\prod_{i=1}^k(\prod_{\alpha,\beta=1}^n(1-z_\alpha^{\pm1}z_\beta^{\pm1}t_i)\prod_{\alpha=1}^n(1-z_\alpha^{\pm1}t_i)^2(1-t_i))}d\nu. \tag{6}$$

In the case of matrices with involution the Poincaré series may be refined to the *-Poincaré series. The algebra with trace generated by k generic matrices together with their transposes may equally well be thought of as the algebra with trace generated by k generic symmetric matrices and k generic skew symmetric matrices. This induces a $2k$-fold grading on the algebra and on the algebra of traces. The corresponding Poincaré series is called the *-Poincaré series. From the point of view of characters, when we pass from a k-fold grading to a $2k$-fold grading, we are passing from a $\mathrm{GL}_k(\mathbb{C})$-character to a $\mathrm{GL}_k(\mathbb{C})\times\mathrm{GL}_k(\mathbb{C})$-character. This theory is discussed in [G].

More generally, given a set of generic $n \times n$ matrices $\{X_\alpha\}$ together with their transposes of some type $\{X_\alpha^*\}$ we may let $S_\alpha = X_\alpha + X_\alpha^*$ and $K_\alpha = X_\alpha - X_\alpha^*$ be generic symmetric and generic skew symmetric matrices, respectively. For fixed k_1 and k_2 let $R = R(n, k_1, k_2; s \text{ or } t)$ be the algebra with trace generated by $S_1, \ldots, S_{k_1}, K_1, \ldots, K_{k_2}$; let $\bar{C} = \bar{C}(n, k_1, k_2; s \text{ or } t)$ be the algebra generated by the traces or elements of R; and let $\bar{R} = \bar{R}(n, k_1, k_2; s \text{ or } t)$ be the algebra generated by R and $\bar{C}$. Also, let F be the commutative algebra generated by the entries of $S_1, \ldots, S_{k_1}, K_1, \ldots, K_{k_2}$. The following is a corollary of Procesi's theorem.

LEMMA 5: *With notation as in the previous paragraph, $F^G = \bar{C}$ and $M_n(F)^G = \bar{R}$.*

Equations (7) and (8) below follow from corollary 7.3 of [B]. Equations (9) and (10), although not explicit in [B], are similar.

$$\bar{C}(2n, k_1, k_2; s)(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}) = \frac{1}{2^n n!} \times$$

$$\oint_T \frac{\prod_{1\le\alpha<\beta\le n}(1 - z_\alpha^{\pm1} z_\beta^{\pm1}) \prod_{\alpha=1}^n (1 - z_\alpha^{\pm2})\, d\nu}{\begin{array}{c}\prod_{i=1}^{k_1}\{\prod_{\alpha,\beta=1}^n (1 - z_\alpha z_\beta^{-1} x_i) \prod_{1\le\alpha<\beta\le n}(1-(z_\alpha z_\beta)^{\pm1} x_i)\} \\ \prod_{i=1}^{k_2}\{\prod_{\alpha,\beta=1}^n (1 - z_\alpha z_\beta^{-1} y_i) \prod_{1\le\alpha\le\beta\le n}(1-(z_\alpha z_\beta)^{\pm1} y_i)\}\end{array}}, \tag{7}$$

$$\bar{R}(2n, k_1, k_2; s)(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}) = \frac{1}{2^n n!} \times$$

$$\oint_T \frac{\prod_{1\le\alpha<\beta\le n}(1 - z_\alpha^{\pm1} z_\beta^{\pm1}) \prod_{\alpha=1}^n (1 - z_\alpha^{\pm2})(\sum_{\alpha,\beta=1}^n z_\alpha + z_\alpha^{-1})^2 d\nu}{\begin{array}{c}\prod_{i=1}^{k_1}\{\prod_{\alpha,\beta=1}^n (1 - z_\alpha z_\beta^{-1} x_i) \prod_{1\le\alpha<\beta\le n}(1-(z_\alpha z_\beta)^{\pm1} x_i) \\ \prod_{i=1}^{k_2}\{\prod_{\alpha,\beta=1}^n (1 - z_\alpha z_\beta^{-1} y_i) \prod_{1\le\alpha\le\beta\le n}(1-(z_\alpha z_\beta)^{\pm1} y_i)\}\end{array}}, \tag{8}$$

$$\bar{C}(2n+1, k_1, k_2; t)(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}) =$$

$$\frac{1}{2^n n!} \prod_{i=1}^{k_1} (1 - x_i)^{-1} \oint_T \frac{\text{numerator}}{\text{denominator}} d\nu, \tag{9}$$

where

$$\text{numerator} = \prod_{1\le\alpha<\beta\le k} (1 - z_\alpha^{\pm1} z_\beta^{\pm1}) \prod_{\alpha=1}^n (1 - z_\alpha^{\pm1})$$

and

$$\text{denominator} = \prod_{i=1}^{k_1}\Big\{\prod_{\alpha,\beta=1}^{n}(1 - z_\alpha z_\beta^{-1}x_i)\prod_{1\le\alpha\le\beta\le n}(1-(z_\alpha z_\beta)^{\pm1}x_i)$$
$$\times\prod_{\alpha=1}^{n}(1 - z_\alpha^{\pm1}x_i)\Big\}\prod_{i=1}^{k_2}\Big\{\prod_{\alpha,\beta=1}^{n}(1 - z_\alpha z_\beta^{-1}y_i)$$
$$\times\prod_{1\le\alpha<\beta\le n}(1-(z_\alpha z_\beta)^{\pm1}y_i)\prod_{\alpha=1}^{n}(1 - z_a^{\pm1}y_i)\Big\},$$

$\bar{R}(2n+1, k_1, k_2; t)(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}) =$

$$\frac{1}{2^n n!}\prod_{\alpha=1}^{k_1}(1-x_i)^{-1}\oint_T\frac{\text{numerator}}{\text{denominator}}d\nu, \tag{10}$$

where

$$\text{numerator} = \prod_{1\le\alpha<\beta\le k}(1 - z_\alpha^{\pm1}z_\beta^{\pm1})\prod_{\alpha=1}^{n}(1-z_\alpha^{\pm1})(1+\sum_{\alpha=1}^{n}(z_\alpha + z_\alpha^{-1})^2)$$

and

$$\text{denominator} = \prod_{i=1}^{k_1}\Big\{\prod_{\alpha,\beta=1}^{n}(1 - z_\alpha z_\beta^{-1}x_i)$$
$$\times\prod_{1\le\alpha\le\beta\le n}(1-(z_\alpha z_\beta)^{\pm1}x_i)\prod_{\alpha=1}^{n}(1 - z_\alpha^{\pm1}x_i)\Big\}$$
$$\prod_{i=1}^{k_2}\Big\{\prod_{\alpha,\beta=1}^{n}(1 - z_\alpha z_\beta^{-1}y_i)\prod_{1\le\alpha<\beta\le n}(1-(z_\alpha z_\beta)^{\pm1}y_i)$$
$$\prod_{\alpha=1}^{n}(1 - z_a^{\pm1}y_i)\Big\}.$$

Aside. The algebras $\bar{C}(n, k_1, k_2; -)$ and $\bar{R}(n, k_1, k_2; -)$ also have an interpretation in terms of G-invariant maps. Instead of all maps from $M_n(\mathbb{C})^k$ to $\mathbb{C}$ or $M_n(\mathbb{C})$, we consider the algebra of G-invariant maps defined on the set of all k_1+k_2-tuples of matrices, the first k_1 of which being constrained to be symmetric and the last k_2 being constrained to be skew symmetric.

2. General properties

A number of general properties of the Poincaré series for generic matrices have been derived from the algebraic properties of these matrices and from properties of complex integrals. Almost all of these arguments can be adapted to our more general setting of matrices with involution, and that is the goal of this section. These properties will be useful in the subsequent computation of the series in individual cases. Our first theorem follows from algebraic considerations.

THEOREM 6: *Each of the Poincaré series and *-Poincaré series for each $\bar{C}$ and $\bar{R}$ (without involution, or with either type of involution) is a rational function. For the Poincaré series, the denominator can be taken to be a product of terms of the form $1 - t^\alpha$ and for the *-Poincaré series case, the denominator can be taken to be a product of terms of the form $1 - x^\alpha y^\beta$.*

Proof: Since the classical groups are all reductive, it follows from the Hochster–Roberts theorem [HR] that $\bar{C}$, their algebras of commutative invariants, are graded Cohen-Macauley. Theorem 6 now follows in the case of $\bar{C}$ from Stanley's theorem, see [S]. For the Poincaré series of $\bar{R}$ we use the observation that the map $f(X_1, \ldots, X_k) \mapsto tr(f(X_1, \ldots, X_k)X_{k+1})$ is a vector space isomorphism of $\bar{R}(n,k)$ with the part of $\bar{C}(n, k+1)$ of degree 1 in X_{k+1}. It follows that

$$P(\bar{R}(n,k)) = \frac{\partial}{\partial x_{k+1}} \bar{C}(n, k+1)|_{x_{k+1}=0}.$$

The *-Poincaré series case is similar. The algebra $\bar{R}(n,k)$ is the vector space direct sum of its symmetric and skew symmetric parts. Let $f(X_1, \ldots, X_k)$ be a symmetric matrix. Then we map it to $tr(f(X_1, \ldots, X_k)S_{k+1})$ in the orthogonal case and to $tr(f(X_1, \ldots, X_k)K_{k+1})$ in the symplectic case. And we do the opposite for skew symmetric matrices. It follows that

$$\bar{R}(n, k_1, k_2; -) = \left.\frac{\partial \bar{C}(k_1+1, k_2; -)}{\partial x_{k_1+1}}\right|_{x_{k_1+1}=0} + \cdot \left.\frac{\partial \bar{C}(k_1, k_2+1; -)}{\partial y_{k_2+1}}\right|_{y_{k_2+1}=0}. \qquad \blacksquare$$

COROLLARY 7: *If $P(\bar{C})$ can be written as a fraction in which $1 - u$ occurs in the denominator with multiplicity a, then $P(\bar{R})$ can be written as a fraction in which $1 - u$ occurs in the denominator with multiplicity at most $a + 1$.*

We have observed that if u is a monomial of degree at least 2, then $1 - u$ occurs in the least denominator of each $P(\bar{C})$ and corresponding $P(\bar{R})$ with equal multiplicity. We conjecture that this will always be the case.

THEOREM 8: *Each Poincaré series is a symmetric function in the t variables, and each *-Poincaré series is a symmetric function in the x variables and in the y variables (separately).*

Proof: Clear. ∎

THEOREM 9: *Let $P(t_1, \ldots, t_k)$ be the Poincaré series for $\bar{C}$, either with or without involution, and with $k \geq 2$, and let $N = 2n$ or $N = 2n+1$ be the size of the matrices. Then P satisfies the functional equation*

$$P(t_1^{-1}, \ldots, t_k^{-1}) = (-1)^d (t_1 \cdots t_k)^{N^2} P(t_1, \ldots, t_k),$$

*where d is $kn+n+1$ in the case of matrices without involution, $k+n$ in the case of transpose-type involution, and n in the case of symplectic involution. Likewise, let $P(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2})$ be the *-Poincaré series for $\bar{C}$, and assume that $k_1 + k_2 \geq 2$; and that $k_2 \geq 1$ in the symplectic case and $k_1 \geq 1$ in the orthogonal case. Then*

$$P(x_1^{-1}, \ldots, x_{k_1}^{-1}, y_1^{-1}, \ldots, y_{k_2}^{-1}) = (-1)^d (x_1 \cdots x_{k_1})^{\alpha} (y_1 \cdots y_{k_2})^{\beta} P(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}),$$

where (d, α, β) is $(n(k_1 + k_2 + 1), 2n^2 - n, 2n^2 + n)$ in the symplectic case and $(n(k_1 + k_2 + 1) + k_1, 2n^2 + 3n + 1, 2n^2 + n)$ in the orthogonal case.

Proof: The case of the ordinary Poincaré series was proven by Teranishi in [T1], and the same proof applies in the *-Poincaré series case. ∎

THEOREM 10: *The *-Poincaré series $P(\bar{C}(2n, k_1, k_2; s))$ and $P(\bar{R}(2n, k_1, k_2; s))$ have poles of order n at each $x_i = 1$ and $y_i = 1$, and the *-Poincaré series $P(\bar{C}(2n+1, k_1, k_2; t))$ and $P(\bar{R}(2n+1, k_1, k_2; t))$ have poles of order n at each $y_i = 1$.*

Proof: In each of the integrals (7), (8), (9) and (10) multiply by $(1 - y_i)^n$ and take the limit as $y_i \to 1$. And in (7) and (8) multiply by $(1 - x_i)^n$ and take the limit as $x_i \to 1$. ∎

3. $\bar{C}$ in the 3×3 case

To help make the concepts of the previous section more clear we reiterate them in the case of 3×3 matrices before trying to describe denominators of the Poincaré series. We will abbreviate $\bar{C}(3,k;t)$, $\bar{R}(3,k;t)$, $\bar{C}(3,k_1,k_2;t)$ and $\bar{R}(3,k_1,k_2;t)$ by $\bar{C}(k)$, etc., suppressing the 3 and the t since this is the only case we deal with in this section.

Up to conjugation there is only one involution on $M_3(\mathbb{C})$. For convenience, we will take the involution to be

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^* = \begin{pmatrix} e & b & h \\ d & a & g \\ f & c & i \end{pmatrix}.$$

The orthogonal group $O(3)$ will consist of those matrices g such that $gg^* = 1$, and $SO(3)$ will consist of matrices g with $gg^* = 1$ and $\det(g) = 1$. The algebra T of all diagonal matrices in $SO(3)$ consists of the matrices of the form

$$g = \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for $z \in \mathbb{C}^\times$. If X_i is a generic matrix, then we calculate

$$g \begin{pmatrix} x_{11}^i & x_{12}^i & x_{13}^i \\ x_{21}^i & x_{22}^i & x_{23}^i \\ x_{31}^i & x_{23}^i & x_{33}^i \end{pmatrix} g^{-1} = \begin{pmatrix} x_{11}^i & z^2 x_{12}^i & z x_{13}^i \\ z^{-2} x_{21}^i & x_{22}^i & z^{-1} x_{23}^i \\ z^{-1} x_{31}^i & z x_{23}^i & x_{33}^i \end{pmatrix}$$

for $g \in T$. It follows that the polynomial algebra $\mathbb{C}[x_{\alpha,\beta}^i]$ has character

$$\prod_i (1-t_i)^{-3}(1-z^2t_i)^{-1}(1-z^{-2}t_i)^{-1}(1-zt_i)^{-2}(1-z^{-1}t_i)^{-2}.$$

Weyl's integration formula now implies that the Poincaré series $P(\bar{C})(t_1,\dots,t_k)$ is equal to

$$\text{(11)} \quad \frac{1}{2(2\pi i)} \oint_{|z|=1} \frac{(1-z)(1-z^{-1})}{\prod_{i=1}^k (1-t_i)^3(1-z^2t_i)(1-z^{-2}t_i)(1-zt_i)^2(1-z^{-1}t_i)^2} \frac{dz}{z}.$$

For $\bar{R}$ (which we will compute in the next section) we need to calculate the action of $g \in T$ on each $x_{\alpha,\beta}^i e_{st}$, where e_{st} is a matrix unit. We leave it to the interested reader to show that this multiplies the numerator by $(1+z+z^{-1})^2$ giving a numerator of $(2-z^3-z^{-3})$.

Our goal is to apply Cauchy's theorem to evaluate the Poincaré series.

This turns out to be somewhat easier to do for the *-Poincaré series. We now calculate the action of a diagonal matrix in SO(3) on a generic symmetric matrix and a generic skew symmetric matrix:

$$g\begin{pmatrix} s^i_{11} & s^i_{12} & s^i_{13} \\ s^i_{21} & s^i_{11} & s^i_{23} \\ s^i_{23} & s^i_{13} & s^i_{33} \end{pmatrix} g^{-1} = \begin{pmatrix} s^i_{11} & z^2 s^i_{12} & z s^i_{13} \\ z^{-2} s^i_{21} & s^i_{11} & z^{-1} s^i_{23} \\ z^{-1} s^i_{23} & z s^i_{13} & z^i_{33} \end{pmatrix},$$

$$g\begin{pmatrix} k^i_{11} & 0 & k^i_{13} \\ 0 & -k^i_{11} & k^i_{23} \\ -k^i_{23} & -k^i_{13} & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} k^i_{11} & 0 & z k^i_{13} \\ 0 & -k^i_{11} & z^{-1} k^i_{23} \\ -z^{-1} k^i_{23} & -z k^i_{13} & 0 \end{pmatrix}.$$

It follows that the character is

$$\prod_{i=1}^{k_1} (1-x_i)^{-2}(1-z^2x_i)^{-1}(1-z^{-2}x_i)^{-1}(1-zx_i)^{-1}(1-z^{-1}x_i)^{-1}$$
$$\times \prod_{i=1}^{k_2} (1-y_i)^{-1}(1-zy_i)^{-1}(1-z^{-1}y_i)^{-1}.$$

We can now use Weyl's integration formula to calculate the *-Poincaré series

$$P(\bar{C}(k_1,k_2))(x_1,\ldots,x_{k_1},y_1,\ldots,y_{k_2}) = \frac{1}{2}(2\pi i)^{-1}\cdot$$

$$\oint_{|z|=1} \frac{(1-z)(1-z^{-1})}{\prod_{i=1}^{k_1}(1-x_i)^2(1-z^2x_i)(1-z^{-2}x_i)(1-zx_i)(1-z^{-1}x_i) \prod_{i=1}^{k_2}(1-y_i)(1-zy_i)(1-z^{-1}y_i)} \frac{dz}{z}. \tag{12}$$

The integrand has simple poles at each $z = x_i$, $z = x_i^{-1}$, $z = \pm x_i^{\frac{1}{2}}$, $z = \pm x_i^{-\frac{1}{2}}$, $z = y_i$ and $z = y_i^{-1}$. If $k_1 = 0$ and $k_2 = 1$ there will also be a pole at $z = 0$. This Poincaré series can be derived from the others and we can disregard it here. Since the x_i and y_i have norm less than 1, the poles inside the unit disk will be at $z = x_i$, $z = \pm x_i^{\frac{1}{2}}$ and $z_i = y_i$. For convenience, we denote the integrand by I,

$$I = \frac{(1-z)(1-z^{-1})}{\prod_{i=1}^{k_1}(1-x_i)^2(1-z^2x_i)(1-z^{-2}x_i)(1-zx_i)(1-z^{-1}x_i) \prod_{i=1}^{k_2}(1-y_i)(1-zy_i)(1-z^{-1}y_i)} \frac{1}{z}.$$

It is now easy to calculate each of the residues. At $z = x_i$ we have

$$\lim_{z\to x_i}(z-x_i)I =$$

$$\frac{(1-x_i)(1-x_i^{-1})}{\prod_{j=1}^{k_1}(1-x_j)^2(1-x_i^2x_j)(1-x_i^{-2}x_j)(1-x_ix_j)\prod_{j\neq i}(1-x_i^{-1}x_j) \prod_{j=1}^{k_2}(1-y_j)(1-x_iy_j)(1-x_i^{-1}y_j)}; \tag{12.1}$$

at $z = \pm x_i^{\frac{1}{2}}$ the residue is

$$\lim_{z \to \pm x_i^{\frac{1}{2}}} (z \mp x_i^{\frac{1}{2}}) I =$$

$$\frac{\frac{1}{2}(1 \mp x_i^{\frac{1}{2}})(1 \mp x_i^{-\frac{1}{2}})}{\prod_{j=1}^{k_1}(1-x_j)^2(1-x_i x_j)(1 \mp x_i^{\frac{1}{2}} x_j)(1 \mp x_i^{-\frac{1}{2}} x_j)\prod_{j\neq i}(1-x_i^{-1}x_j) \prod_{j=1}^{k_2}(1-y_j)(1\mp x_i^{\frac{1}{2}} y_j)(1 \mp x_i^{-\frac{1}{2}} y_j)}; \tag{12.2}$$

and, finally, at $z = y_i$ we have the residue

$$\lim_{z \to y_i} (z - y_i) I =$$

$$\frac{(1-y_i)(1-y_i^{-1})}{\prod_{j=1}^{k_1}(1-x_j)^2(1-y_i^2 x_j)(1-y_i^{-2}x_j)(1-y_i x_j)(1-y_i^{-1}x_j) \prod_{j=1}^{k_2}(1-y_j)(1-y_i y_j)\prod_{j\neq i}(1-y_i^{-1}y_j)}. \tag{12.3}$$

So, the Poincaré series of $\bar{C}(k_1, k_2)$ is the sum of these fractions. In practice, this sum may be very difficult to do. However, Theorem 6 implies that the sum will be a rational function in the x_i and y_i; and that the denominator will divide a product of terms of the form $(1-m)$, where m will be a monomial in the x_i's and y_j's. We can use this information to calculate a denominator and to prove that it is minimal.

COROLLARY 11: *$P(\bar{C}(k_1, k_2))$ can be expressed as a rational function with denominator*

$$\prod_{i=1}^{k_1}(1-x_i)(1-x_i^2)(1-x_i^3) \prod_{1 \le i < j \le k_1}(1-x_i x_j)(1-x_i^2 x_j)(1-x_i x_j^2)$$

$$\prod_{1 \le i \le j \le k_2}(1-y_i y_j)\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}(1-x_i y_j)(1-x_i y_j^2).$$

Proof: By Theorem 6, we need only consider the factors in the denominators of (12.1), (12.2) and (12.3) which divide (1− a monic polynomial). At the $z = x_i$ residue, we exclude $(1-x_i^{-2}x_j)$ and $(1-x_i^{-1}x_j)$ for $i \neq j$. At the $z = \pm x_i^{\frac{1}{2}}$ residue we rationalize the denominator to replace the factor of $(1 \mp x_i^{\frac{1}{2}} x_j)(1 \mp x_i^{-\frac{1}{2}} x_j)$ by $(1-x_i x_j^2)(1-x_i^{-1}x_j^2)$, and the factors of $(1 \mp x_i^{\frac{1}{2}} y_j)(1 \mp x_i^{-\frac{1}{2}} y_j)$ by $(1-x_i y_j^2)(1-x_i^{-1}y_j^2)$. By Theorem 6, we may then ignore the terms $(1-x_i^{-1}x_j^2)$, $(1-x_i^{-1}y_j^2)$, and $(1-x_i^{-1}x_j)$ for $i \neq j$. Finally, at the $z = y_i$ residue terms, we may ignore each $(1-y_i^{-2}x_j)$, $(1-y_i^{-1}x_j)$ and $(1-y_i^{-1}y_j)$. Also, in these terms note that each y_i has a pole of order one at each of 1 and -1. Taking the least common denominator of the remaining terms gives the desired denominator. ∎

LEMMA 12:
(a) $P(\bar{C}(2,0)) =$

$$\frac{1+x_1^2x_2^2+x_1^4x_2^4}{(1-x_1)(1-x_1^2)(1-x_1^3)(1-x_2)(1-x_2^2)(1-x_2^3)(1-x_1x_2)(1-x_1^2x_2)(1-x_1x_2^2)}.$$

(b) $P(\bar{C}(1,1)) = \dfrac{1-x_1y_1+x_1^2y_1^2}{(1-x_1)(1-x_1^2)(1-x_1^3)(1-y_1^2)(1-x_1y_1)(1-x_1y_1^2)}$.

(c) $P(\bar{C}(0,2)) = \dfrac{1}{(1-y_1^2)(1-y_2^2)(1-y_1y_2)}$.

Proof: Computation. ∎

THEOREM 13: *The *-Poincaré series $P(\bar{C}(k_1,k_2))$ can be written as a fraction in lowest terms with denominator*

$$\prod_{i=1}^{k_1}(1-x_i)(1-x_i^2)(1-x_i^3)\prod_{1\le i<j\le k_1}(1-x_ix_j)(1-x_i^2x_j)(1-x_ix_j^2)$$

$$\prod_{1\le i\le j\le k_2}(1-y_iy_j)\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}(1-x_iy_j)(1-x_iy_j^2).$$

Proof: By Corollary 11, we know that this polynomial is a denominator, and by Lemma 12 we know that it is a least denominator in the cases of $(k_1,k_2)=(2,0)$, $(1,1)$ or $(0,2)$. By specializing one of the variables to zero in Lemma 12, we also know that the theorem is true in the case of $(k_1,k_2)=(1,0)$ or $(0,1)$. Now, in the general case, the denominator must be a symmetric polynomial and it must divide the given one. But, if some factor was not present, we could specialize some of the variables to zero (depending on which term was not present) to get a smaller denominator in one of the cases (1,0), (0,1), (2,0), (1,1) or (0,2). ∎

We can derive a similar theorem for ordinary Poincaré series $P(\bar{C}(k))$ using Theorem 13.

THEOREM 14: *The Poincaré series $P(\bar{C}(k))$ can be written as a fraction in lowest terms with denominator*

$$\prod_{i=1}^{k}(1-t_i)(1-t_i^2)^3(1-t_i^3)^2\prod_{1\le i<j\le k}(1-t_it_j)^3(1-t_it_j^2)^2(1-t_i^2t_j)^2.$$

Proof: The *-Poincaré series and the Poincaré series are related by

$$P(\bar{C}(k))(t_1,\ldots,t_k) = P(\bar{C}(k,k))(t_1,\ldots,t_k;t_1,\ldots,t_k).$$

Hence, it follows from the previous theorem that the polynomial

$$\prod_{i=1}^{k}(1-t_i)(1-t_i^2)^3(1-t_i^3)^2 \prod_{1\le i<j\le k}(1-t_it_j)^4(1-t_it_j^2)^2(1-t_i^2t_j)^2$$

is a denominator for $P(\bar{C}(k))$. This agrees with the denominator we are proving except for the exponent of $(1-t_it_j)$. So, consider the integral (11). The poles inside the unit disk are at each $z=t_i$ and $z=\pm t_i^{\frac{1}{2}}$. The latter are simple poles and only contribute $(1-t_it_j)$ to the first power. However, $z=t_i$ is a pole of order two, so in order to calculate the residue we must first multiply the integrand by $(z-t_i)^2$, take the partial derivative with respect to t_i and then set $z=t_i$. Taking the partial derivative and using the product rule will cause the $(1-t_iz)^2$ in the denominator to be replaced by $(1-t_jz)^3$, and after substitution this will become $(1-t_jt_i)^3$.

This shows that the polynomial in the statement is a denominator for $P(\bar{C}(k))$. In order to prove that it is a least denominator, as in the previous theorem, it suffices to prove that it is a least denominator in the case $k=2$. The relevant integral can be evaluated using Macsyma. The denominator is as claimed. The numerator is quite messy (over 100 terms), but it can be verified that it has no factors in common with this denominator. ∎

What can we say about the numerators? Let $n(-)(t_1,\ldots,t_k)$ be the numerator of $P(-)(t_1,\ldots,t_k)$ in lowest terms. Then our computation of the denominators in this section lets us use the functional equations for the Poincaré series from Theorem 9 to get functional equations for the numerators.

THEOREM 15: *If each Poincaré series is written as a rational function with denominator as in Theorems 13 and 14, then the numerators satisfy functional equations*

(1) $n(\bar{C}(k))(t_1^{-1},\ldots,t_k^{-1}) = \pm(t_1\cdots t_k)^{7-11k}n(\bar{C}(k))(t_1,\ldots,t_k)$. *In particular, it is of degree* $11k-7$ *in each* t_i *and total degree* $k(11k-7)$. *The term of highest degree is* $\pm(t_1\cdots t_k)^{11k-7}$.

(2) $n(\bar{C}(k_1,k_2))(x_1^{-1},\ldots,x_{k_1}^{-1},y_1^{-1},\ldots,y_{k_2}^{-1}) = \pm(x_1\cdots x_{k_1})^{4-4k_1-2k_2}\times (y_1\cdots y_{k_2})^{2-3k_1-k_2}n(\bar{C}(k_1,k_2))(x_1,\ldots,x_{k_1},y_1,\ldots,y_{k_2})$ *when* $k_1\ge 1$. *In particular, it is of degree* $4k_1+2k_2-4$ *in each* x_1 *and* $3k_1+k_2-2$ *in each* y_i *and total degree* $k_1(4k_1+2k_2-4)+k_2(3k_1+k_2-2)$. *The term of highest degree is* $\pm(x_1\cdots x_{k_1})^{4k_1+2k_2-4}(y_1\cdots y_{k_2})^{3k_1+k_2-2}$.

(3) $n(\bar{C}(0,k_2))(y_1^{-1},\dots,y_{k_2}) = \pm(y_1,\dots,y_{k_2})^{2-k_2} n(\bar{C}(0,k_2))(y_1,\dots,y_{k_2})$ *when* $k_2 \geq 2$. *In particular, it is of degree* $k_2 - 2$ *in each* y_i *and total degree* $k_2(k_2-2)$. *The term of highest degree is* $\pm(y_1\cdots y_{k_2})^{k_2-2}$.

Proof: Each numerator is the product of the denominator times the Poincaré series. Theorems 13 and 14 imply that the denominators satisfy functional equations of the given type. Theorem 9 implies that the Poincaré series satisfy functional equations except for the $k_2 = 0$ case. However, since the Poincaré series equals the sum of (12.1), (12.2) and (12.3), it must satisfy

$$P(x_1^{-1},\dots,x_{k_1}^{-1},y_1^{-1},\dots,y_{k_2}^{-1}) = (-1)^{k_2+1}(x_1\cdots x_{k_1})^6(y_1\cdots y_{k_2})^3 P(x_1,\dots,x_{k_1},y_1,\dots,y_{k_2}).$$

The functional equations for the numerators follow. As for the degrees, each Poincaré series is monic and each denominator is monic, so the numerator must be monic. The degree now follows from the functional equation. ∎

4. $\bar{R}$ in the 3×3 case

The case of $P(\bar{R})$ is similar. In this case

$$P(\bar{R}(k_1,k_2)) = \frac{1}{2}(2\pi i)^{-1}\times$$

$$\oint_{|z|=1} \frac{(1-z)(1-z^{-1})(1+z+z^{-1})^2}{\prod_{i=1}^{k_1}(1-x_i)^2(1-z^2x_i)(1-z^{-2}x_i)(1-zx_i)(1-z^{-1}x_i)\prod_{i=1}^{k_2}(1-y_i)(1-zy_i)(1-z^{-1}y_i)}\frac{dz}{z}. \tag{13}$$

If $3k_1 + k_2 \geq 4$ we avoid poles at $z=0$ and, again, the only poles inside the unit disk will be at $z = x_i$, $z = \pm x^{\frac{1}{2}}$ and $z = y_i$. So, the residues will be:

At $z = x_i$:

$$\frac{(1-x_i)(1-x_i^{-1})(1+x_i+x_i^{-1})^2}{\prod_{j=1}^{k_1}(1-x_j)^2(1-x_i^2x_j)(1-x_i^{-2}x_j)(1-x_ix_j)\prod_{j\neq i}(1-x_i^{-1}x_j)\prod_{j=1}^{k_2}(1-y_j)(1-x_iy_j)(1-x_i^{-1}y_j)}. \tag{13.1}$$

At $z = \pm x_i^{\frac{1}{2}}$:

$$\frac{(1\mp x_i^{\frac{1}{2}})(1\mp x_i^{-\frac{1}{2}})(1\pm x^{\frac{1}{2}}\pm x_i^{-\frac{1}{2}})^2}{\prod_{j=1}^{k_1}(1-x_j)^2(1-x_ix_j)(1\mp x_i^{\frac{1}{2}}x_j)(1\mp x_i^{-\frac{1}{2}}x_j)^2\prod_{j\neq i}(1-x_i^{-1}x_j)\prod_{j=1}^{k_2}(1-y_j)(1\mp x_i^{\frac{1}{2}}y_j)(1\mp x_i^{-\frac{1}{2}}y_j)}. \tag{13.2}$$

And at $z = y_i$:

$$\frac{(1-y_i)(1-y_i^{-1})(1+y_i+y_i^{-1})^2}{\substack{\prod_{j=1}^{k_1}(1-x_j)^2(1-y_i^2x_j)(1-y_i^{-2}x_j)(1-y_ix_j)(1-y_i^{-1}x_j)\\ \prod_{j=1}^{k_2}(1-y_j)(1-y_iy_j)\prod_{j\neq i}(1-y_i^{-1}y_j)}}. \tag{13.3}$$

Examining the cases the reader will notice that, in (13.1) there is a factor of $(1+x_i+x_i^2)$ in the numerator which reduces the factor of $(1-x_i^3)$ in the denominator to $(1-x_i)$; and in (13.2) there is a factor of $(1\pm x_i^{\frac{1}{2}}+x_i)$ in the numerator which reduces the factor of $(1-x_i^{\frac{3}{2}})$ in the denominator to $(1\mp x_i^{\frac{1}{2}})$. Taking these cancellations into account gives this analogue of Corollary 11:

COROLLARY 16: *$P(\bar{R}(k_1,k_2))$ can be expressed as a rational function with denominator*

$$\prod_{i=1}^{k_1}(1-x_i)^2(1-x_i^2)\prod_{1\le i<j\le k_1}(1-x_ix_j)(1-x_i^2x_j)(1-x_ix_j^2)$$
$$\prod_{1\le i\le j\le k_2}(1-y_iy_j)\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}(1-x_iy_j)(1-x_iy_j^2).$$

Here is the analogue of Lemma 12:

LEMMA 17:

(a) $P(\bar{R}(2,0)) =$

$$\frac{1+x_1x_2+x_1^2x_2^2}{(1-x_1)^2(1-x_1^2)(1-x_2)^2(1-x_2^2)(1-x_1x_2)(1-x_1^2x_2)(1-x_1x_2^2)}.$$

(b) $P(\bar{R}(1,1)) = \dfrac{(1+y_1+y_1^2)}{(1-x_1)^2(1-x_1^2)(1-y_1^2)(1-x_1y_1)(1-x_1y_1^2)}.$

(c) $P(\bar{R}(0,2)) = \dfrac{1+y_1+y_2+y_1^2+2y_1y_2+y_2^2+y_1^2y_2+y_1y_2^2}{(1-y_1^2)(1-y_2^2)(1-y_1y_2)}.$

Combining these two results gives Theorem 18, and Theorem 19 follows with a bit more work.

THEOREM 18: *The $*$-Poincaré series $P(\bar{R}(k_1,k_2))$ can be written as a fraction in lowest terms with denominator*

$$\prod_{i=1}^{k_1}(1-x_i)^2(1-x_i^2)\prod_{1\le i<j\le k_1}(1-x_ix_j)(1-x_i^2x_j)(1-x_ix_j^2)$$
$$\prod_{1\le i\le j\le k_2}(1-y_iy_j)\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}(1-x_iy_j)(1-x_iy_j^2).$$

THEOREM 19: *The Poincaré series $P(\bar{R}(k))$ can be written as a fraction in lowest terms with denominator*

$$\prod_{i=1}^{k}(1-t_i)^3(1-t_i^2)^3 \prod_{1\le i<j\le k}(1-t_it_j)^3(1-t_it_j^2)^2(1-t_i^2t_j)^2.$$

Proof: Taking $k_1=k_2=k$ and letting each x_i and y_i equal t_i, it follows from Theorem 18 that $P(\bar{R}(k))$ can be written as a fraction with denominator

$$\prod_{i=1}^{k}(1-t_i)^2(1-t_i^2)^3(1-t_i^3) \prod_{1\le i<j\le k}(1-t_it_j)^4(1-t_it_j^2)^2(1-t_i^2t_j)^2.$$

So, we need to eliminate the extra factors of $(1-t_i^3)$ and $(1-t_it_j)$. The latter can be handled as in Theorem 10. As for the $(1-t_i^3)$, refer to (13.1), (13.2) and (13.3). If we try to specialize each x_i and y_i to t_i before adding, there is no problem with (13.2), but (13.1) has a factor of $(1-x_i^{-1}y_i)=x_i^{-1}(x_i-y_i)$ in the denominator and (13.3) has a factor of $(1-y_i^{-1}x_i)=y_i^{-1}(y_i-x_i)$, each of which would become zero. Since these are the only terms which vanish if x_i and y_i are set equal, it follows that if we add these two terms there will be a factor of (x_i-y_i) in the numerator which will cancel this term in the denominator of the sum.

Now, if we add (13.1) and (13.3) the denominator will have a factor of $(1-y_i^2x_i)$ which specializes to $(1-t_i^3)$, which cannot be cancelled before specialization. This is the only such term and the source of the $(1-t_i^3)$ that we need to deal with. However, the numerator of the sum will be of the form $(1+x_i+x_i^2)A+(1+y_i+y_i^2)B$ and on specialization it has a factor of $(1+t_i+t_i^2)$, turning the unwanted $(1-t_i^3)$ in the denominator into the extra factor of $(1-t_i)$ we need.

Finally, a Macsyma computation shows that this denominator is correct in the case of $k=2$ which shows that it is least possible in general. ∎

In the general case we do not know $P(\bar{C}(k))$ or $P(\bar{R}(k))$ because we do not know the numerators. In the case of $k_1=0$ where all of the matrices are skew symmetric, we can get a complete description of $P(\bar{C}(0,k))$ and $P(\bar{R}(0,k))$ using Schur functions.

THEOREM 20: (a) $P(\bar{C}(0,k))=\sum_{a,b,c\ge 0} S_{(2a+2b+c,2b+c,c)}(y_1,\ldots,y_k)$.

(b) $P(\bar{R}(0,k))=$

$$\sum_{a,b,c\ge 0} S_{(2a+2b+c,2b+c,c)}(y_1,\ldots,y_k)+\sum_{\lambda_1>\lambda_2\ge\lambda_3} S_{(\lambda_1,\lambda_2,\lambda_3)}(y_1,\ldots,y_k)$$
$$+\sum_{\lambda_1\ge\lambda_2\ge\lambda_3,\ \lambda_2\ge 1} S_{(\lambda_1,\lambda_2,\lambda_3)}(y_1,\ldots,y_k).$$

Proof: (a) The integral (12) reduces to

$$\oint_{|z|=1} \frac{(1-z)(1-z^{-1})}{\prod_{i=1}^{k}(1-y_i)(1-y_i z)(1-y_i z^{-1})}\frac{dz}{z} = \prod_{i=1}^{k}(1-y_i)^{-1}\oint_{|z|=1}\frac{(1-z)(1-z^{-1})}{\prod_{i=1}^{k}(1-y_i z)(1-y_i z^{-1})}\frac{dz}{z}.$$

This latter integral can be expanded using Cauchy's identity (cf. [FuH], A.13)

$$\frac{(1-z)(1-z^{-1})}{\prod_{i=1}^{k}(1-y_i z)(1-y_i z^{-1})} = (1-z)(1-z^{-1})\sum_{\lambda} S_\lambda(z,z^{-1})S_\lambda(y_1,\ldots,y_k).$$

But, the Schur function $S_\lambda(z,z^{-1})$ is zero unless λ has height at most two. Hence, the integral can be expanded as a series in Schur function of height at most two, and the coefficients can be calculated from the $k=2$ case, which we calculated in Lemma 12(c). It equals

$$(1+y_1)^{-1}(1+y_2)^{-1}(1-y_1y_2)^{-1} = \sum_i (-1)^i S_i(y_i,y_2)\sum_j S_{j,j}(y_1,y_2) = \sum_{i,j}(-1)^i S_{i+j,j}(y_1,y_2).$$

To get from this sum to $P(\bar{C}(0,k))$ we need to multiply by $\prod_i(1-y_i)^{-1} = \sum_m S_m$. By Young's rule, the coefficient of S_λ, $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ equals

$$\sum_{\mu=(\mu_1,\mu_2)\subseteq\lambda}(-1)^{\mu_1-\mu_2}.$$

Now, $\mu\subseteq\lambda$ if and only if $\lambda_2\le\mu_1\le\lambda_1$ and $\lambda_3\le\mu_2\le\lambda_2$. Hence the above sum equals

$$\sum_{\mu_1=\lambda_2}^{\lambda_1}\sum_{\mu_2=\lambda_3}^{\lambda_2}(-1)^{\mu_1-\mu_2} = \sum_{\mu_1=\lambda_2}^{\lambda_1}(-1)^{\mu_1}\sum_{\mu_2=\lambda_3}^{\lambda_2}(-1)^{\mu_2}.$$

This sum will be zero if either $\lambda_1-\lambda_2$ or $\lambda_2-\lambda_3$ is odd. If they are both even, it will equal $(-1)^{\lambda_3}(-1)^{\lambda_2}$, which will equal 1 since $\lambda_2-\lambda_3$ is even.

(b) We need to evaluate Lemma 12(c) in terms of Schur functions for $k=2$. Again, we pull out the factor of $(1-y_1)^{-1}(1-y_2)^{-1}$. Now

$$\frac{1+y_1+y_2+y_1^2+2y_1y_2+y_2^2+y_1^2y_2+y_1y_2^2}{(1+y_1)(1+y_2)(1-y_1y_2)} = \frac{1}{(1+y_1)(1+y_2)(1-y_1y_2)}+\frac{y_1+y_2}{(1-y_1y_2)}.$$

The first term is $\sum_{i,j}(-1)^i S_{i+j,j}(y_1,y_2)$, as in part (a). The second term is $\sum_i S_{i+1,i}(y_1,y_2)$. The rest of the proof is an exercise in Young's rule. ∎

Aside. The proof of Theorem 15 shows that in order to calculate $P(3,0,k;s)$ for all k it suffices to calculate $P(\bar{C}(3,0,2;t))$ and that the general case follows from properties of Schur functions. Likewise for $\bar{R}$. This generalizes easily. The ordinary Poincaré series $P(\bar{C}(2n+1,k;t))$ or $P(\bar{R}(2n+1,k;t))$ is completely determined by the case $k=4n^2+2n$, and $P(\bar{C}(2n,k;s))$ and $P(\bar{R}(2n,k;s))$ are determined by the case $k=4n^2-2n$; and the *-Poincaré series would be determined by $(k_1,k_2)=(2n^2+2n,2n^2)$ in the transpose case and $(k_1,k_2)=(2n^2-2n,2n^2)$ in the symplectic case.

Here is the analogue of Theorem 15 for $\bar{R}$.

THEOREM 21: *If each Poincaré series is written as a rational function with denominator as in Theorems 18 and 19, then the numerators satisfy functional equations*

(1) $n(\bar{R}(k))(t_1^{-1},\dots,t_k^{-1})=\pm(t_1\cdots t_k)^{9-9k}n(\bar{R}(k))(t_1,\dots,t_k)$. *In particular, it is of degree* $9k-9$ *in each* t_i *and total degree* $k(9k-9)$. *The term of highest degree is* $\pm(t_1\cdots t_k)^{9k-9}$.

(2) $n(\bar{R}(k_1,k_2))(x_1^{-1},\dots,x_{k_1}^{-1},y_1^{-1},\dots,y_{k_2}^{-1}) = \pm(x_1\cdots x_{k_1})^{6-4k_1-2k_2}\times (y_1\cdots y_{k_2})^{2-3k_1-k_2}n(\bar{R}(k_1,k_2))(x_1,\dots,x_{k_1},y_1,\dots,y_{k_2})$ *when* $k_1\geq 1$. *In particular, it is of degree* $4k_1+2k_2-6$ *in each* x_1 *and* $3k_1+k_2-2$ *in each* y_i *and total degree* $k_1(4k_1+2k_2-6)+k_2(3k_1+k_2-2)$. *The term of highest degree is* $\pm(x_1\cdots x_{k_1})^{4k_1+2k_2-6}(y_1\cdots y_{k_2})^{3k_1+k_2-2}$.

(3) $n(\bar{C}(0,k_2))(y_1^{-1},\dots,y_{k_2}) = \pm(y_1,\dots,y_{k_2})^{2-k_2}n(\bar{C}(0,k_2))(y_1,\dots,y_{k_2})$ *if* $k_2\geq 2$. *In particular, it is of degree* k_2-2 *in each* y_i *and total degree* $k_2(k_2-2)$. *The term of highest degree is* $\pm(y_1\cdots y_{k_2})^{k_2-2}$.

5. 4 × 4 matrices with symplectic involution

We would like to imitate the arguments of the previous section for the case of 4×4 matrices with symplectic involution. The main obstacle is that the computations are much longer. For example, to calculate $P(\bar{C}(k_1,k_2))$ (again suppressing the size of the matrices and the type of involution from the notation) we need to evaluate the integral

$$\frac{1}{8}(2\pi i)^{-2}\prod(1-x_i)^{-2}\prod(1-y_i)^{-2}\oint\frac{\text{numerator}}{\text{denominator}}\frac{dz_2}{z_2}\wedge\frac{dz_1}{z_1}, \tag{15}$$

where

$$\begin{aligned}\text{numerator} =&(1 - z_1z_2)(1 - z_1z_2^{-1})(1 - z_1^{-1}z_2)(1 - z_1^{-1}z_2^{-1})\\ &\times (1 - z_1^2)(1 - z_1^{-2})(1 - z_2^2)(1 - z_2^{-2})\end{aligned}$$

and

$$\begin{aligned}\text{denominator} =& \prod_{i=1}^{k_1}(1 - z_1z_2x_i)(1 - z_1z_2^{-1}x_i)(1 - z_1^{-1}z_2x_i)(1 - z_1^{-1}z_2^{-1}x_i)\\ &\times \prod_{i=1}^{k_2}(1 - z_1z_2y_i)(1 - z_1z_2^{-1}y_i)(1 - z_1^{-1}z_2y_i)(1 - z_1^{-1}z_2^{-1}y_i)\\ &\times (1 - z_1^2y_i)(1 - z_1^{-2}y_i)(1 - z_2^2y_i)(1 - z_2^{-2}y_i).\end{aligned}$$

It seems strange, but rather than evaluate (15) directly, it turns out to be easier to evaluate

$$\text{(16)}\qquad \frac{1}{8}(2\pi i)^{-2}\prod(1 - x_i)^{-2}\prod(1 - y_i)^{-2}\oint \frac{\text{numerator}}{\text{denominator}}\frac{dz_2}{z_2}\wedge\frac{dz_1}{z_1}$$

where the numerator is as in (15), but the denominator is given by

$$\begin{aligned}&\text{denominator} =\\ &\qquad\prod(1 - z_1z_2a_i)\prod(1 - z_1z_2^{-1}b_i)\prod(1 - z_1^{-1}z_2c_i)\prod(1 - z_1^{-1}z_2^{-1}d_i)\\ \text{(16)}\qquad &\qquad\times\prod(1 - z_1^2e_i)\prod(1 - z_1^{-2}f_i)\prod(1 - z_2^2g_i)\prod(1 - z_2^{-2}h_i),\end{aligned}$$

where we have replaced the x's and y's in the integral with independent variables a through h, all assumed to be less than 1 in absolute value. A similar device was used by Van den Bergh in his study of the case of matrices without involution in [V]. Information obtained from (16) can be transferred to (15) by specializing the new variables back to x's and y's. (So, for example, some of the a_i would specialize to x's and some to y's.) To simplify the computation of (16) we will assume

$$\text{(17)}\qquad b_i < \sqrt{h_j} < d_k \quad \text{for all } i, j \text{ and } k.$$

Since the result of (16) is a rational function, it certainly suffices to identify it under this restriction.

LEMMA 22: *If there are no poles at zero, then the integral (16) can be evaluated by adding the residues at the poles* $(z_1, z_2) = (\sqrt{f_i}, b_j\sqrt{f_i})$, $(\sqrt{c_id_j}, \sqrt{d_j/c_i})$, $(\sqrt{f_i}, d_j/\sqrt{f_i})$, $(d_i\sqrt{g_j}, 1/\sqrt{g_j})$, $(c_i\sqrt{h_j}, \sqrt{h_j})$ *and* $(\sqrt{f_i}, \sqrt{h_j})$.

Proof: In the inner integral z_2 has poles inside the unit disk at $b_i z_1$, $d_i z_1^{-1}$ and $\sqrt{h_i}$. Taking first $z_2 = b_i z_1$, if we multiply the integrand by $z_2 - b_i z_1$ and take the limit as z_2 approaches $b_i z_1$ we get a fraction with denominator

$$\prod_j (1 - b_i a_j z_1^2) \prod_{j \neq i} (1 - b_i^{-1} b_j) \prod_j (1 - b_i c_j) \prod_j (b_i^{-1} d_j z_1^{-2})$$
$$\prod_j (1 - e_j z_1^2) \prod_j (1 - f_j z_1^{-2}) \prod_j (1 - b_i^2 g_j z_1^2) \prod_j (1 - b_i^{-2} h_j z_1^{-2}).$$

The poles for z_1 in the unit disk would be

$$\sqrt{d_j / b_i}, \ \sqrt{f_j}, \text{ and } \sqrt{h_j}/b_i,$$

but thanks to assumption (17) only $\sqrt{f_j}$ is possible. In this case $z_2 = b_i \sqrt{f_j}$.

Next, the residue at $z_2 = d_i z_1^{-1}$ has denominator

$$\prod_j (1 - a_j d_i) \prod_j (1 - b_j d_i^{-1} z_1^2) \prod_j (1 - c_j d_i z_1^{-2}) \prod_{j \neq i} (1 - d_i^{-1} d_j)$$
$$\prod_j (1 - e_j z_1^2) \prod_j (1 - f_j z_1^{-2}) \prod_j (1 - g_j d_i^2 z_1^{-2}) \prod_j (1 - h_j d_i^{-2} z_1^2).$$

Taking into account (17), there are three types of poles in the disk: $z_1 = \sqrt{c_j d_i}$, $z_1 = \sqrt{f_j}$ and $z_1 = d_i \sqrt{g_j}$. The respective z_2 values are $\sqrt{d_i / c_j}$, $d_i / \sqrt{f_j}$ and $1/\sqrt{g_j}$.

Finally, in the case of $z_2 = \sqrt{h_i}$ the residue has denominator

$$\prod_j (1 - a_j \sqrt{h_i} z_1) \prod_j (1 - b_j / \sqrt{h_i} z_1) \prod_j (1 - c_j \sqrt{h_i} z_1^{-1}) \prod_j (1 - d_j / \sqrt{h_i} z_1^{-1})$$
$$\prod_j (1 - e_j z_1^2) \prod_j (1 - f_j z_1^{-2}) \prod_j (1 - g_j h_i) \prod_{j \neq i} (h_j h_i^{-1}).$$

This creates poles in the disk at $z_1 = c_j \sqrt{h_i}$ and $z_1 = \sqrt{f_j}$. This completes the proof, except for the technical remark that one should also consider the terms with the square root replaced by the negative square root. In all cases this will not effect the resulting residues and so we will ignore this point. ∎

It is easy to evaluate the residues in the lemma: Simply substitute z_1 and z_2 by the indicated values, deleting the two terms which would become zero in the denominator.

LEMMA 23: *The $*$-Poincaré series $P(\bar{C}(k_1, k_2))$ for the trace ring of 4×4 matrices with symplectic involution can be written as a rational function with denominator*

$$\prod_{\substack{j_1\neq j_2,\ j_3\\ j_2<j_3}} (1-y_{j_1}^2 y_{j_2} y_{j_3}) \prod_{j_1\leq j_2} (1-y_{j_1}^2 y_{j_2}^2) \prod_{j_1\neq j_2} (1-y_{j_1}^3 y_{j_2})$$
$$\prod_{j_1\leq j_2} (1-x_i^2 y_{j_1} y_{j_2})(1-x_i y_{j_1} y_{j_2}) \prod_{i_1<i_2} (1-x_{i_1} x_{i_2} y_j)$$
$$\prod_{j_1<j_2<j_3} (1-y_{j_1} y_{j_2} y_{j_3}) \prod_{j_1\neq j_2} (1-y_{j_1}^2 y_{j_2}) \prod_{j_1\leq j_2} (1-y_{j_1} y_{j_2})$$
$$\prod_{i_1\leq i_2} (1-x_{i_1} x_{i_2}) \prod_i (1-x_i^2),$$

where the x indices are understood to go from 1 to k_1 and the y indices from 1 to k_2.

Proof: By Theorem 6 we know that the result of the computation will be a rational function with denominator a product of terms which divide a monic polynomial, and so we may ignore all terms in each summand not of this form, confident in the knowledge that they will cancel in the sum. If we merely take the least common denominator of the remaining terms, the result will not be as claimed in the lemma. There would be an extra factor of $\prod(1-x_i)$. However, Theorem 10 controls the order of the pole at $x_i = 1$ and guarantees that this term will cancel.

It would be quite tedious to read the complete computation of the denominator. We will just show how to compute one of the terms as an example and the interested reader may check the rest. If $(z_1, z_2) = (\sqrt{c_i d_j}, \sqrt{d_j/c_i})$, then when we specialize we may let c_i or d_j be either an x or a y. There are therefore four possible terms to consider. (Other cases will require fewer possibilities since the variables e, f, g and h can only be specialized to y's.) Because we will want to refer to it later, we will do the case in which $c_i = x_I$ and $d_j = x_J$. In this case the numerator will become

$$(1-x_J)(1-x_J^{-1})(1-x_I)(1-x_I)$$
$$(1-x_I x_J)(1-x_I^{-1} x_J^{-1})(1-x_J x_I^{-1})(1-x_J^{-1} x_I).$$

Note that if $I = J$, then the numerator becomes zero and so we may assume in this case that $I \neq J$. The denominator is

$$\prod_i (1-x_J x_i)(1-x_I x_i) \prod_{i\neq I} (1-x_I^{-1} x_i) \prod_{i\neq J} (1-x_J^{-1} x_i)$$

$$\prod_j (1-x_Jy_j)(1-x_J^{-1}y_j)(1-x_Iy_j)(1-x_I^{-1}y_j)$$
$$(1-x_Ix_Jy_j)(1-x_I^{-1}x_J^{-1}y_j)(1-x_Jx_I^{-1}y_j)(1-x_J^{-1}x_Iy_j).$$

The terms not involving inverses are $(1-x_Jx_i)$, $(1-x_Ix_i)$ and $(1-x_Ix_Jy_j)$. Since $I \neq J$ each $(1-x_{i_1}x_{i_2})$ will occur at most once in the product. If i is I or J then there will cancellation with the $(1-x_I)$ or $(1-x_J)$ in the numerator. Finally, taking into account the $\prod_i(1-x_i)^{-2}\prod_j(1-y_j)^{-2}$ outside of the integral, the contribution to the denominator is

$$\prod_i(1-x_i)\prod_j(1-y_j)^2\prod_{i_1\le i_2}(1-x_{i_1}x_{i_2})\prod_{i_1<i_2}\prod_j(1-x_{i_1}x_{i_2}y_j).$$

This divides the denominator asserted by the lemma. The remaining cases offer few additional subtleties. ∎

COROLLARY 24: *$P(\bar{C}(k,0))$, the Poincaré series for the trace ring of the ring generated by k symmetric 4×4 matrices with symplectic involution, can be written as a fraction in lowest terms with denominator $\prod_i(1-x_i)\prod_{i\le j}(1-x_ix_j)$.*

Proof: In the proof of the previous lemma, if there are no y's then we need not consider any terms containing e's, f's, g's or h's. And there is only one pole which doesn't: The one we actually computed in the proof of the previous lemma. The denominator there is the one claimed. To see that it is a least denominator we may calculate the Poincaré series in the case of $k=2$. We get

$$P(\bar{C}(2,0)) = (1-x_1)^{-1}(1-x_2)^{-1}(1-x_1^2)^{-1}(1-x_2^2)^{-1}(1-x_1x_2)^{-1}$$

and this completes the proof. ∎

We don't have the computational power to verify that the denominator of Lemma 22 is a least denominator in general. In order to do this, one would need at least to calculate the cases of $(k_1,k_2)=(0,3),(1,2)$ and $(2,1)$. Here are some cases we did calculate:

LEMMA 25:
(a) $P(\bar{C}(2,0)) = (1-x_1x_2)^{-1}(1-x_1^2)^{-1}(1-x_2^2)^{-1}(1-x_1)^{-1}(1-x_2)^{-1}$.
(b) $P(\bar{C}(1,1)) = (1-x^2y^2)^{-1}(1-y^4)^{-1}(1-xy^2)^{-1}(1-y^2)^{-1}(1-x^2)^{-1}(1-x)^{-1}$.
(c) $P(\bar{C}(0,2)) =$

$$\frac{1-y_1^2y_2-y_1y_2^2+y_1^2y_2^2+y_1^4y_2^2+2y_1^3y_2^3+y_1^2y_2^4-y_1^4y_2^5-y_1^5y_2^4+y_1^6y_2^6}{\begin{array}{c}(1-y_1^2y_2^2)(1-y_1^3y_2)(1-y_1y_2^3)(1-y_1^4)(1-y_2^4)(1-y_1^2y_2)(1-y_1y_2^2)\\(1-y_1y_2)(1-y_1^2)(1-y_2^2)\end{array}}.$$

Note that the denominator (c) is in agreement with Lemma 22 and that (b) has a factor of $(1+x)$ missing.

To do the case of $\bar{R}$ instead of $\bar{C}$ we need to multiply (15), and therefore (16), by $(z_1+z_2+z_1^{-1}+z_2^{-1})^2$, which changes the numerator to

$$(1-z_1^2z_2^2)(1-z_1^2z_2^{-2})(1-z_1^{-2}z_2^2)(1-z_1^{-2}z_2^{-2})(1-z_1^2)(1-z_1^{-2})(1-z_2^2)(1-z_2^{-2}).$$

It turns out that this tends to make all of the computations a bit easier by causing more terms in the denominator to cancel. Lemma 22 still holds in this case. Here are the analogues of Lemma 23, Corollary 24 and Lemma 25:

LEMMA 26: *The $*$-Poincaré series $P(\bar{R}(k_1,k_2))$ for 4×4 matrices with symplectic involution and trace can be written as a rational function with denominator*

$$\prod_{\substack{j_1\neq j_2\ j_3\\ j_2<j_3}}(1-y_{j_1}^2y_{j_2}y_{j_3})\prod_{j_1\neq j_2}(1-y_{j_1}^3y_{j_2})\prod_{j_1<j_2}(1-x_i^2y_{j_1}y_{j_2})$$
$$\prod_{j_1\leq j_2}(1-x_iy_{j_1}y_{j_2})\prod_{i_1<i_2}(1-x_{i_1}x_{i_2}y_j)\prod_{j_1<j_2<j_3}(1-y_{j_1}y_{j_2}y_{j_3})$$
$$\prod_{j_1\neq j_2}(1-y_{j_1}^2y_{j_2})\prod_{j_1<j_2}(1-y_{j_1}y_{j_2})^2$$
$$\prod_{i_1<i_2}(1-x_{i_1}x_{i_2})\prod_{i,j}(1-x_iy_j)\prod_i(1-x_i)^2\prod_j(1-y_j^2)(1-y_j),$$

where the x indices are understood to go from 1 to k_1 and the y indices from 1 to k_2.

COROLLARY 27: *$P(\bar{R}(k,0))$, the Poincaré series for the ring with trace generated by k generic symmetric matrices, is a rational function with least denominator $\prod_{i=1}^k(1-x_i)^2\prod_{i<j}(1-x_ix_j)$.*

LEMMA 28: (a) $P(\bar{R}(2,0))=(1-x_1)^{-2}(1-x_2)^{-2}(1-x_1x_2)^{-1}$.

(b) $P(\bar{R}(1,1))=(1-x)^{-2}(1-y)^{-1}(1-y^2)^{-1}(1-xy)^{-1}(1-xy^2)^{-1}$.

(c) $P(\bar{R}(0,2))=$

$$\frac{1+y_1^2y_2^2}{(1-y_1)(1-y_2)(1-y_1^2)(1-y_2^2)(1-y_1y_2)^2(1-y_1^2y_2)(1-y_1y_2^2)(1-y_1^3y_2)(1-y_1y_2^3)}.$$

(d) $P(\bar{R}(2,1)) = (1-x_1y^2)^{-1}(1-x_2y^2)^{-1}(1-x_1x_2y)^{-1}(1-x_1x_2)^{-1}\times(1-x_1y)^{-1}(1-x_2y)^{-1}(1-y^2)^{-1}(1-y)^{-1}(1-x_1)^{-2}(1-x_2)^{-2}$.

Note that the results in Lemma 28 agree with Lemma 26. We go so far as to base a conjecture on this.

CONJECTURE 29: *If $k_2 \neq 0$ then $P(\bar{C}(k_1,k_2))$ can be written as a fraction in lowest terms with denominator*

$$\prod_{\substack{j_1\neq j_2\ j_3\\ j_2<j_3}}(1-y_{j_1}^2y_{j_2}y_{j_3})\prod_{j_1\leq j_2}(1-y_{j_1}^2y_{j_2}^2)\prod_{j_1\neq j_2}(1-y_{j_1}^3y_{j_2})$$
$$\prod_{j_1\leq j_2}(1-x_i^2y_{j_1}y_{j_2})(1-x_iy_{j_1}y_{j_2})\prod_{i_1<i_2}(1-x_{i_1}x_{i_2}y_j)$$
$$\prod_{j_1<j_2<j_3}(1-y_{j_1}y_{j_2}y_{j_3})\prod_{j_1\neq j_2}(1-y_{j_1}^2y_{j_2})\prod_{j_1\leq j_2}(1-y_{j_1}y_{j_2})$$
$$\prod_{i_1\leq i_2}(1-x_{i_1}x_{i_2})\prod_i(1-x_i),$$

and $P(\bar{R}(k_1,k_2))$ can be written as a fraction in lowest terms with denominator

$$\prod_{\substack{j_1\neq j_2\ j_3\\ j_2<j_3}}(1-y_{j_1}^2y_{j_2}y_{j_3})\prod_{j_1\neq j_2}(1-y_{j_1}^3y_{j_2})\prod_{j_1<j_2}(1-x_i^2y_{j_1}y_{j_2})$$
$$\prod_{j_1\leq j_2}(1-x_iy_{j_1}y_{j_2})\prod_{i_1<i_2}(1-x_{i_1}x_{i_2}y_j)\prod_{j_1<j_2<j_3}(1-y_{j_1}y_{j_2}y_{j_3})$$
$$\prod_{j_1\neq j_2}(1-y_{j_1}^2y_{j_2})\prod_{j_1<j_2}(1-y_{j_1}y_{j_2})^2$$
$$\prod_{i_1<i_2}(1-x_{i_1}x_{i_2})\prod_{i,j}(1-x_iy_j)\prod_i(1-x_i)^2\prod_j(1-y_j^2)(1-y_j).$$

If in $P(\bar{C}(k,k))$ and $P(\bar{R}(k,k))$ we set each x_i and each y_i equal to t_i, we get $P(\bar{C}(k))$ and $P(\bar{R}(k))$, respectively. These will be rational functions and, using Lemmas 23 and 26, we can calculate denominators. We presume that these denominators will not be minimal, but that they won't be far off. In order to express the result more compactly, we introduce some notation. Given a k-tuple α, let A be the set of all permutations of α. Define $[\alpha] := \prod_{\beta\in A}(1-t^\beta)$. For example, if $\alpha=(2,1,0,\ldots,0)$, which we will abbreviate as $(2,1)$, then $[2,1]=\prod_{1\leq i\neq j\leq k}(1-t_i^2t_j)$; and if $\alpha=(1,1)$, then $[1,1]=\prod_{1\leq i<j\leq k}(1-t_it_j)$.

THEOREM 30: *The Poincaré series $P(\bar{C}(k))$ can be written as a fraction with denominator*

$$[4]^2[3,1]^2[2,2]^3[2,1,1]^2[3][2,1]^4[1,1,1]^7[2]^3[1,1]^2$$

and $P(\bar{R}(k))$ can be written as a fraction with denominator

$$[3,1]^2[2,1,1]^2[3][2,1]^4[1,1,1]^7[2]^2[1,1]^5[1]^3.$$

We conclude by proving an analogue of Theorem 20 and express the Poincaré series of $\bar{C}(k,0)$ and $\bar{R}(k,0)$ in terms of Schur functions. Two preliminaries are needed.

LEMMA 31: *The Poincaré series for each of $\bar{C}(k,0)$ and $\bar{R}(k,0)$ can be written as $\prod(1-x_i)^2$ times a series $\sum m_\lambda S_\lambda$ in the Schur functions, in which each λ has height at most 4. In particular, the coefficients m_λ are determined by the $k=4$ case.*

Proof: In each case, the Poincaré series is the integral of a fraction, with numerator a function of z_1 and z_2, and whose denominator is

$$\prod_{i=1}^{k}(1-x_i)^2(1-z_1z_2x_i)(1-z_1z_2^{-1}x_i)(1-z_1^{-1}z_2x_i)(1-z_1^{-1}z_2^{-1}x_i).$$

By the Cauchy identity, the integrand is the numerator times

$$\prod_i(1-x_i)^{-2}\sum_\lambda S_\lambda(z_1z_2,z_1z_2^{-1},z_1^{-1}z_2,z_1^{-1}z_2^{-1})S_\lambda(x_1,\ldots,x_k).$$

If the height of λ is greater than 4, the first Schur function will be zero, and the lemma follows. ∎

LEMMA 32: *The Poincaré series for $\bar{C}(k,0)$ for $k\geq 4$ and the Poincaré series for $\bar{R}(k,0)$ for $k\geq 5$ satisfy the functional equation*

$$P(x_1^{-1},\ldots,x_k^{-1})=(x_1\cdots x_k)^6P(x_1,\ldots,x_k).$$

Proof: Consider the case of $\bar{C}$. The Poincaré series is given by the integral

$$\frac{1}{8}\prod_{i=1}^{k}(1-x_i)^{-2}\oint_T\frac{\text{numerator}}{\text{denominator}}\,d\nu$$

where, as in (15), the numerator is

$$\begin{aligned}\text{numerator}=&(1-z_1z_2)(1-z_1z_2^{-1})(1-z_1^{-1}z_2)(1-z_1^{-1}z_2^{-1})\\&\times(1-z_1^2)(1-z_1^{-2})(1-z_2^2)(1-z_2^{-2})\end{aligned}$$

and the denominator is

$$\text{denominator}=\prod_{i=1}^{k}(1-z_1z_2x_i)(1-z_1z_2^{-1}x_i)(1-z_1^{-1}z_2x_i)(1-z_1^{-1}z_2^{-1}x_i).$$

There is a pole of order 4 at $z_1 = 0$ in the numerator and of order $2k-1$ (the 1 coming from the $d\nu$) in the denominator. Hence, under the hypothesis $k \geq 4$, the fraction does not have a pole at $z_1 = 0$. So, to evaluate the integral we need consider only the poles at $z_1 = x_I z_2$ and at $z_1 = x_I/z_2$. The two cases are similar, so we will consider only the former. In this case the residue is a fraction with

$$\begin{aligned}\text{numerator} =&(1-x_Iz_2^2)(1-x_I)(1-x_I^{-1})(1-x_I^{-1}z_2^{-2})\\ &\times(1-x_I^2z_2^2)(1-x_I^{-2}z_2^{-2})(1-z_2^2)(1-z_2^{-2})\end{aligned}$$

and

$$\begin{aligned}\text{denominator} =&\prod_{i\neq I}(1-x_Ix_iz_2^2)(1-x_Ix_i)(1-x_I^{-1}x_i)(1-x_I^{-1}x_iz_2^{-2})\\ &\times(1-x_I^2z_2^2)(1-x_I^2)(1-z_2^{-2}).\end{aligned}$$

Note that the $(1-z_2^{-2})$ in the numerator and denominator cancel, so there are no poles at $z_2 = \pm 1$. Taking this cancellation into account, the pole at $z_2 = 0$ is of order 5 in the numerator and order $2k-3$ in the denominator. Hence, for $k \geq 4$ the fraction has no pole at $z_2 = 0$. So, the only poles will be at $z_2 = \pm\sqrt{x_J/x_I}$ for J such that $x_I > x_J$. The residue will be the fraction with

$$\begin{aligned}\text{numerator} =&(1-x_J)(1-x_I)(1-x_I^{-1})(1-x_J^{-1})\\ &\times(1-x_Ix_J)(1-x_I^{-1}x_J^{-1})(1-x_J/x_I)(1-x_I/x_J)\end{aligned}$$

and

$$\begin{aligned}\text{denominator} =&\prod_{i\neq I,J}(1-x_Jx_i)(1-x_Ix_i)(1-x_I^{-1}x_i)(1-x_J/x_i)\\ &\times(1-x_Ix_J)(1-x_I^2)(1-x_J/x_I)\\ &\times 2(1-x_J)^2(1-x_Ix_J)(1-x_J/x_I).\end{aligned}$$

We leave it to the reader to check that each such term satisfies the functional equation claimed.

The case of $\bar{R}$ is similar, with the higher value of k needed because of the extra factor of $(\sum_{\alpha=1}^{2} z_\alpha + z_\alpha^{-1})^2$ in the numerator. ∎

THEOREM 33: (a) *Given a and b, define* even(a,b) *to be the number of even integers n, $a \leq n \leq b$ and* odd(a,b) *to be the number of odds. Then the Poincaré series of $\bar{C}(k,0)$ is given by $\sum_{ht(\lambda)\leq 6} m_\lambda S_\lambda$, where the coefficient m_λ is given by*

$$m_\lambda = \prod_{i=1}^{5}\text{even}(\lambda_{i+1},\lambda_i) + \prod_{i=1}^{5}\text{odd}(\lambda_{i+1},\lambda_i).$$

(b) *The Poincaré series of* $\bar{R}(k,0)$ *is given by* $\sum_{ht(\lambda)\leq 6} m_\lambda S_\lambda$, *where the coefficient* m_λ *is given by*

$$m_\lambda = (\lambda_1-\lambda_2+1)(\lambda_2-\lambda_3+1)(\lambda_3-\lambda_4+1)(\lambda_4-\lambda_5+1)(\lambda_5-\lambda_6+1).$$

Proof: (a) By Corollary 24, the Poincaré series is

$$f(x_1,\dots,x_k)\prod_i(1-x_i)^{-2}\prod_i(1+x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1},$$

for some polynomial f. It follows from Lemma 32 that for $k\geq 4$, the numerator f is of degree $n-4$. Hence, if $n=4$, the numerator is a constant, and it is not hard to see that it must be 1. So,

$$P(x_1,\dots,x_4)=\prod_i(1-x_i)^{-2}\prod_i(1+x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}.$$

Applying Lemma 30, we turn to the computation of the product

$$\prod_i(1+x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}$$

in terms of Schur functions. It follows from [M], example 4, section I.5, that this equals

$$\sum_{\mathrm{ht}(\lambda)\leq 4}(-1)^{|\lambda|}S_\lambda,$$

and we need to multiply this by $\prod_i(1-x_i)^{-2}=(\sum_i S_{(i)})^2$. We multiply by the two factors of $\sum_i S_{(i)}$ one at a time. By Young's rule, the coefficient of each S_μ, height $\mu\leq 5$, in the product $\sum_i S_{(i)}\sum_\lambda(-1)^{|\lambda|}S_\lambda$ is

$$\sum_{\mu_2\leq\nu_1\leq\mu_1}(-1)^{\nu_1}\times\sum_{\mu_3\leq\nu_2\leq\mu_2}(-1)^{\nu_2}\times\cdots\times\sum_{\mu_5\leq\nu_4\leq\mu_4}(-1)^{\nu_4}.$$

Each sum is 0 or 1, depending on whether the difference $\mu_i-\mu_{i-1}$ is odd or even, respectively. Hence, the product is the sum $\sum S_\mu$, summed over all μ of height at most 5 in which each difference $\mu_i-\mu_{i-1}$ is even. So, either every part of μ is even, or every part is odd. We apply Young's rule to the product

$$\sum_i S_{(i)}\sum_{\mu=(2a+e,2b+e,2c+e,2d+e,e)}S_\mu.$$

The coefficient of S_λ in such a sum will be the number of μ of height at most 5 such that each μ_i is between λ_{i+1} and λ_i, and such that either all parts of μ are even or all parts are odd. Formula (a) follows.

(b) The second case is similar and a bit easier. In this case the functional equation of Lemma 32 implies that the Poincaré series in five variables is $\prod_i(1-x)^{-2}\prod_{i<j}(1-x_ix_j)^{-1}$. It follows from [M], example 5(b), section I.5, and from Lemma 31 that we need to compute $(\sum_i S_{(i)})^2\sum_{a\geq b}S_{(a,a,b,b)}$. It follows from Young's rule that

$$\sum_i S_{(i)}\sum_{a\geq b}S_{(a,a,b,b)} = \sum_{\mathrm{ht}(\lambda)\leq 5} S_\lambda,$$

and also that

$$\sum_i S_{(i)}\sum_{\mathrm{ht}(\lambda)\leq 5} S_\lambda$$

is as claimed. ∎

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